

Courant Institute of
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Magneto-Fluid Dynamics Division

Boltzmann Collision Operator with
Inverse-Power Intermolecular Potentials

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Preface

This report studies the linearized collision operator in the Boltzmann equation with repulsive intermolecular potentials $V(r) = ar^{-\alpha}$.

It is shown that for $\alpha > 2$ the collision operator has a purely discrete spectrum and its eigenfunctions are infinitely differentiable $\mathcal{L}^2$ -functions which are complete in $\mathcal{L}^2$. The proof relies on the formalism of pseudo-differential operators.

The report contains two parts: Part I deals with the special case of $\alpha = 4$, the Maxwell's molecules, while Part II considers the general case of inverse-power potentials with $\alpha > 2$.

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Part I

Maxwell's Molecules -- The Inverse-Fourth Power Potential

1. Introduction

The present study is concerned with the linearized Boltzmann equation [5]

$$\frac{Df}{Dt} = C[f] \quad (1)$$

$$C[f] = \int d^3c_1 \exp(-c_1^2) \int d\epsilon d\theta |g| I(|g|, \theta) [f'_1 + f' - f_1 - f] \quad (2)$$

in the kinetic theory of gases with repulsive intermolecular potentials

$$V(r) = a_1/r^\alpha . \quad (3)$$

It is convenient for our purpose to introduce a new function

$$\phi(c) = f(c) \exp(-\frac{1}{2}c^2) \quad (4)$$

and consider the operator

$$L\phi = -\exp(-\frac{1}{2}c^2) C[\phi(c) \exp(\frac{1}{2}c^2)] . \quad (5)$$

which is known to be symmetric and non-negative.

We shall show that for $\alpha > 2$, L^{-1} is a compact operator from $\mathcal{X}^2$ to $\mathcal{X}^2$, excluding the zero-eigenvalue subspace. Consequently the spectrum of L is purely discrete and its eigenfunctions are $\mathcal{X}^2$ forming a complete set in $\mathcal{X}^2$.

In presenting these results, we divide the study into two parts. The first part, which constitutes the present paper, deals with the special case of $\alpha = 4$, commonly known

as Maxwell's molecular model or Maxwell's gas, while the second part, given in a companion paper [8] (hereafter referred to as Part II), considers the general case of inverse-power potentials with $\alpha > 2$.

The result for the Maxwell's gas ($\alpha = 4$) is well-known from the work of Wang Chang and Uhlenbeck [11], in which the eigenfunctions are explicitly constructed. However, our method, which is capable of obtaining the most crucial theoretical results regarding the discreteness of the spectrum and the differentiability and decay of the eigenfunctions, does not depend on the explicit construction of the eigenfunctions and can therefore be carried over to the general case of inverse-power potentials with $\alpha > 2$.

A brief description of our method can be given as follows. We first formulate the operator

$$A\phi = (N+L)\phi \quad (6)$$

in the form of a pseudo-differential operator, where N is a large positive number. Since this operator is difficult to handle, we further construct a new operator

$$A_\ell \phi = (N+L_\ell)\phi \quad (7)$$

from (6) which has the property that $A\phi = A_\ell \phi$ if ϕ has the form

$$\phi(c) = Y_\ell(c, z)\psi(|c|) \quad (8)$$

where $Y_\ell(c, z)$ is the ℓ -th order spherical harmonics of the

vector c with z as polar axis. It should be noted that since L is isotropic, all eigenfunctions of A must be of the form (8).

Our result then follows from the proof that A_ℓ is invertible and A_ℓ^{-1} is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$, $\mathcal{L}^2$ being the square-integrable functions, and $\|A_\ell^{-1}\| \rightarrow 0$ as $\ell \rightarrow \infty$.

The basic features of our method can best be illustrated with the special case of Maxwell molecules, since this special case is not overburdened with complicated computations and lengthy estimates that are inevitable to the general case of inverse-power potentials. At the same time, the result we shall obtain in this paper for the Maxwell molecules provides the key to the case of general inverse-power potentials. This is so because for large c the collision operator with an inverse-power potential for $\alpha \neq 4$ can be approximated in a certain sense by an operator which is very similar to the collision operator for the Maxwell's gas (i.e. with a potential for $\alpha = 4$). It is for these reasons that we shall first present herewith the result for the Maxwell's gas in an independent part, separate from the general case in Part II for $\alpha > 2$.

In the next section a general discussion is given for the background of the problem concerning inverse-power law Boltzmann equation. Section 3 provides some material from the theory of pseudo-differential operators which is needed for our analyses in this Part I as well as in Part II. In section 4, the collision operator for the Maxwell's gas is

formulated as a pseudo-differential operator. The operator A_ℓ , called the associated pseudo-differential operator of A , is constructed in Sec. 5. The main result of this paper is contained in Sec. 6 where we prove that A_ℓ is invertible and A_ℓ^{-1} is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$, and $\|A_\ell^{-1}\| \rightarrow 0$ as $\ell \rightarrow \infty$. This shows that A^{-1} is compact from $\mathcal{L}^2$ to $\mathcal{L}^2$, or equivalently, L^{-1} is compact from $\mathcal{L}_\perp^2$ to $\mathcal{L}_\perp^2$ where $\mathcal{L}_\perp^2$ are the square-integrable functions orthogonal to the five eigenfunctions of L with the zero-eigenvalue. From this we then conclude that the spectrum of L is discrete and the eigenfunction exist and form a complete orthonormal set of $\mathcal{L}^2$ -functions. Further computations show that the eigenfunctions are actually in $\mathcal{S}$. But this will not be done here, since we shall present a proof for this fact in the companion paper [8] (i.e. Part II) for the general case of inverse-power potentials with $\alpha > 2$.

2. The Background

For the molecular model of hard spheres, the differential cross-section $I(|g|, \theta)$ is finite and the collision operator $L\phi$ in (5) is well-understood, at least qualitatively [3],[6]. For all other practical physical models [5], such as the inverse-power law model, the Lenard-Jones' model, the Sutherland's model, etc., the interparticle potential has an infinite range and consequently the differential cross-section has a non-integrable singularity in θ at $\theta = \frac{1}{2}\pi$. Because of this singularity, the collision operator $L\phi$ is a non-standard unbounded operator and very little is known about the nature of this operator. The only exception in this connection is the work of Wang Chang and Uhlenbeck [11] in which the eigenfunctions of $L\phi$ are explicitly constructed for the inverse fourth-power potential model (the Maxwell gas). But their method of explicit construction has not been possible to be generalized to other infinite-ranged potentials or other inverse-power law potentials and a general understanding of the collision operator for infinite-range potentials is completely lacking.

Several ways have been used or suggested to avoid the difficulty caused by the singularity of $I(|g|, \theta)$. One is to use the angle cut-off approximation [6] in which $I(|g|, \theta)$ is set to be zero in a small interval $\frac{1}{2}\pi - \theta > \epsilon$ near $\frac{1}{2}\pi$. Another approach uses the radial cut-off approximation [4] in which the interparticle potential $V(r)$ is set to vanish for $r > R$ for a large R . Still another approach [13] suggests

the use of quantum mechanical (instead of classical) cross-section for $I(|g|, \theta)$ which is finite. This is a method to avoid the singularity of $I(|g|, \theta)$ by quantum mechanical cross-section for the classical gas (at reasonably high temperatures). It should be distinguished from the quantum mechanical calculation of a low-temperature gas for which quantum mechanical cross-section must be used.

In all the above approaches, the differential cross-section is finite and equation (2) can be separated into two terms as

$$C[f] = -v(c)f(c) + K[f] , \quad (9)$$

$$K[f] = \int K(c, \eta) f(\eta) d^3\eta . \quad (10)$$

for which a somewhat standard theory can be developed.

However, in the above approaches, both $v(c)$ and $K(c, \eta)$ become infinite as the parameter ϵ , and $1/R$ and $\hbar$ (the Planck constant) tend to zero. This shows that we have to carry the two terms in (9) which both tend to infinity while on physical ground the final results of the theory must remain independent of the parameters ϵ , $1/R$ and $\hbar$ as they become vanishingly small. This clearly indicates that these approaches cannot lead to a satisfactory theory of classical gases. Even from the viewpoint of numerical analysis, the form given by (9) as the difference of two extremely large terms would not be suitable.

We then conclude from the above consideration that to develop a satisfactory theory of classical gases, the

difficulty arising from the singularity of $I(g, \theta)$ cannot and should not be avoided. The purpose of the present study is to develop a theory which is capable of handling the Boltzmann collision operator $L\phi$ with $I(|g|, \theta)$ having a non-integrable singularity in θ .

In studying the collision operator, we are confronted with the alternatives of working with (1) the eigenvalue problem for $L\phi$, (2) initial value problems (including the sound propagation problem) of the Boltzmann equation, or (3) simple boundary value problems of the Boltzmann equation. We have decided to choose the eigenvalue problem since it is the simplest mathematically well-defined problem while the remaining two types of problems are more difficult and therefore probably need some approximations which we do not desire to go into at this stage. At any rate, the eigenvalue problem seems to be worthwhile to be looked into and it is hoped the understanding of the eigenvalue problem will be useful to more difficult problems of the Boltzmann equation.

3. Some Preliminaries

This section gives a simple discussion of pseudo-differential operators (PsDO) which will be sufficient for our need in this Part I as well as Part II.

We shall throughout this paper use c and k to denote three-dimensional vectors while their scalars are denoted as $|c|$ and $|k|$. We shall also use the shorthand notation

$$\partial_c^m = \partial_{c_1}^{m_1} \partial_{c_2}^{m_2} \partial_{c_3}^{m_3}$$

$$|m| = m_1 + m_2 + m_3$$

where c_i , $i=1,2,3$ are the three components of c . Similar notation will be used for the k -derivatives.

An operator P defined by

$$P\phi = \int e^{ik \cdot c} p(c,k) \hat{\phi}(k) dk \quad (11)$$

is called a PsDO if $p(c,k)$, called the symbol of the operator P , satisfies

$$|\partial_c^m \partial_k^n p(c,k)| < a_{mn} (1+c^2)^\sigma (1+k^2)^\mu \quad (12)$$

for $|m| \geq 0$ and $|n| \geq 0$ where σ and μ may be positive and $\hat{\phi}$ is the Fourier transform

$$\hat{\phi}(k) = (2\pi)^{-3} \int e^{-ik \cdot c} \phi(c) dc. \quad (13)$$

In the case of Boltzmann collision operator, the indices σ and μ in (12) are positive and this makes this class of PsDO difficult to treat since differentiation with c and k

does not bring down the growth in c and k ; this is essentially the difficulty that leads us to consider the operator A_ρ in (7) in lieu of the operator A in (6). However, for positive σ and μ , the following result can easily be obtained which is all we need in dealing with the operator A in (6).

THEOREM 1. The operator P in (11) with condition (12) maps $\mathcal{S}$ into $\mathcal{S}$.

The proof follows from the form

$$P\phi = \int e^{ik \cdot c} (1 - \partial_k^2)^j \{p_1(c, k) \psi(k)\} dk \quad (13)$$

$$p_1(c, k) = p(c, k) (1 + c^2)^{-j} (1 + k^2)^{-\ell} \quad (14)$$

$$\psi(k) = (1 + k^2)^\ell \hat{\phi}(k) \quad (15)$$

where $p_1(c, k)$ and its derivatives are square-integrable over $R^3 \times R^3$. This completes the proof.

We can show that P maps $\mathcal{S}$ into $\mathcal{L}^2$ if (12) only holds for $m=0$ and $0 \leq n_i \leq n^*$, n^* is the nearest integer for $n^* \geq \sigma+2$. On the other hand, the operator A_ρ in (7) leads to a much nicer class of PsDO. In this connection, our analyses rely on two theorems due to Calderón and Vaillancourt [1,2]. One of their theorems may be stated as follows.

THEOREM 2. If $p(c, k)$ satisfies (12) for $0 \leq n_i \leq 3$, $0 \leq m_i \leq 3$, $i=1,2,3$ and with $\sigma = \mu = 0$, then the operator P in (11) from $\mathcal{S}$ to $\mathcal{L}^2$ can be extended to a bounded operator from

$\mathcal{L}^2$ to $\mathcal{L}^2$. Furthermore the $\mathcal{L}^2$ -norm $\|P\|$ of P satisfies

$$\|P\| < b \max |a_{mn}| \quad (16)$$

where the maximum is taken over $0 \leq m_i, n_i \leq 3$ and the constant b is independent of $p(c, k)$.

For the proof of this theorem, we refer to Ref. 1.

An immediate consequence of Theorem 2 is that P is compact for $\sigma < 0$, $\mu < 0$. This may be stated as follows.

Theorem 3. If $p(c, k)$ satisfies (12) with $0 \leq m_i, n_i \leq 3$, $i=1, 2, 3$, for $\sigma < 0$, $\mu < 0$, then P in (11) is a compact operator on $\mathcal{L}^2$.

To prove this, it suffices to consider

$$p_n(c, k) = \psi_n(c, k)p(c, k)$$

$$P_n \phi = \int e^{ik \cdot c} p_n(c, k) \hat{\phi}(k) d^3 k$$

where $\psi_n(c, k)$ is a C^∞ -function having the property $\psi_n(c, k) \equiv 1$ for $c^2 + k^2 < n$ and $\psi_n(c, k) \equiv 0$ for $c^2 + k^2 > 2n$, $n=1, 2, \dots$. Then P_n are compact operators and

$$\|P - P_n\| \rightarrow 0$$

as $n \rightarrow \infty$, according to Theorem 2. Therefore P is compact.

The other theorem of Calderón and Vaillancourt that we have to use concerns PsDO with multiple symbols. For our purpose, it is sufficient to introduce these operators as

$$P\phi = \int \int e^{ik \cdot (c-\eta)} p(c, \eta, k) \phi(\eta) d^3\eta d^3k \quad (17)$$

where $p(c, \eta, k)$, called a multiple symbol, satisfies

$$|\partial_c^{\ell} \partial_{\eta}^m \partial_k^n p(c, \eta, k)| < p_0 (1+k^2)^{-\mu}, \quad \mu > 0 \quad (18)$$

for all n_i, m_i, ℓ_i such that $0 \leq n_i \leq 4$, $0 \leq |m| \leq 8$, and $0 \leq |\ell| \leq 8$. It is not difficult to see that P , defined by (17) and (18), maps $\mathcal{S}$ into $\mathcal{L}^2$. In fact, we can get an extra $(1+\eta^2)^{-2}$ factor in the integrand by using $(1+\eta^2)^2 \phi \in \mathcal{S}$, an extra factor $(1+k^2)^{-2}$ at the expense of differentiation with respect to η and an extra factor $(1+c^2)^{-1}$ by the Peetre inequality

$$(1+\eta^2)^{-1} < 2(1+c^2)^{-1}(1+|c-\eta|^2)$$

where the factor $(1+|c-\eta|^2)$ can be compensated at the expense of differentiation with respect to k . The end result is that $P\phi$ becomes a sum of terms each of the form

$$\int \int e^{ik \cdot (c-\eta)} [1+k^2]^{-1} \tilde{p}(c, \eta, k) \psi(\eta) d^3\eta d^3k$$

with $\tilde{p}$ a square-integrable function of $R^3 \times R^3 \times R^3$ and ψ square-integrable in η . This shows that $P\phi \in \mathcal{L}^2$.

We now state the second theorem of Calderón and Vaillancourt [2] in the following simplified version.

THEOREM 4. The operator P , defined by (17) and (18) with $\mu > 0$, can be extended to a bounded operator from $\mathcal{L}^2$ to $\mathcal{L}^2$

and furthermore

$$\|P\| < bp_0 ,$$

where p_0 is the constant in (18) and the constant b is independent of $p(c,\eta,k)$.

For the proof of this theorem we refer to Ref. 2. It is only mentioned that the extra factor $(1+k^2)^{-\mu}$ in (18) enables us to avoid the necessity of approximating $p(c,\eta,k)$ by p_j of compact support in k as was used in Ref. 2.

4. The Collision Operator (with $\alpha = 4$)

For inverse-power law potentials, the scattering cross-section has the simple form

$$|g| I(|g|, \theta) = |g|^{1-4/\alpha} \beta(\theta, \alpha) \sin \theta \quad (19)$$

and for the Maxwell's gas, $\alpha = 4$ and consequently

$$|g| I(|g|, \theta) = \beta(\theta, 4) \sin \theta \quad (20)$$

which is independent of g .

We now substitute

$$\phi(c) = \int e^{ik \cdot c} \hat{\phi}(k) d^3 k \quad (21)$$

into (5) and obtain

$$L\phi = \int e^{ik \cdot c} p(c, k) \hat{\phi}(k) d^3 k \quad (22)$$

$$p(c, k) = -\exp\left\{-\frac{1}{2}c^2 - ik \cdot c\right\} C\left[\exp\left\{\frac{1}{2}c^2 + ik \cdot c\right\}\right] \quad (23)$$

which is the PsDO form, where the operator C is given in (2).

It turns out that some of the integrations in (23) can be carried out explicitly due to the simplification in (20).

The result is

$$p(c, k) = q(c, k) + r(c, k) \quad (24)$$

$$q(c, k) = \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin \theta [1 - H(\cos \theta)] d\theta \quad (25)$$

$$r(c, k) = \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin \theta \left[(2)^{\frac{3}{2}} \exp\left\{-\frac{1}{2}c^2 - \frac{1}{2}k^2 - ik \cdot c\right\} - H(\sin \theta) \right] d\theta \quad (26)$$

where the function H is

$$H(x) = \left[\frac{2}{(2-x^2)} \right]^{\frac{3}{2}} J_0 \left[\frac{\sin 2\theta |c \times k|}{(2-x^2)} \right] \exp \left[\frac{-\frac{1}{2}x^2(c^2+k^2+2ik \cdot c)}{(2-x^2)} \right] \quad (27)$$

and J_0 is the zero-th order Bessel function.

The derivation of (24), (25) and (26) proceeds as follows.

We write (23) as

$$p(c,k,\mu) = \exp \left[\frac{i}{2}(c+\mu k)^2 \right] C \left[\exp \frac{i}{2}(c+\mu k)^2 \right] \quad (28)$$

where μ is a complex variable and we note that $p(c,k,\mu=i) = p(c,k)$. But first we consider (28) for real μ . For real μ , (28) can be integrated but by using the method of Waldmann [10] with his λ taken to be $-\frac{1}{2}$. We then observe that for fixed c and k the result so obtained is an entire function of μ . On the other hand, for fixed c and k , the right-hand side of (28) is also an entire function of μ , as can be seen from (2). Therefore we can use analytic continuation and set $\mu=i$ in the resultant formula of Waldmann [10] from which (24), (25) and (26) immediately follows.

The behavior of $p(c,k)$ can be studied by expressing the dominant contribution near $\theta = \frac{1}{2}\pi$ in (25) in terms of confluent hypergeometric functions, which we shall not do since we shall not have to use this in our computations. Suffice it to say that $p(c,k)$ is $C^\infty(R^3 \times R^3)$ and satisfies condition (12) in Sec. 3 with $\sigma = \mu = \frac{1}{\alpha} = \frac{1}{4}$. This establishes the fact the L maps ϕ into ϕ .

As we mentioned in Sec. 3, that condition (12) with σ and μ positive makes the operator (in this case, $N+L$) difficult to treat. The difficulty, arising from the fact that differentiation with respect to c does not bring down the growth in c in (12), is caused by the term $c \times k$ in (25). The next section shows how to get around this obstacle.

Lastly, we note that the singularity of $\beta(\theta, \alpha)$ at $\theta = \frac{1}{2} \pi$ may be expressed as

$$\beta(\theta, \alpha) = \beta_0 (\cos \theta)^{-1-2\nu} + B(\theta),$$

$$|B(\theta)| \leq \beta_1 (1 + \tan^2 \theta)^\nu,$$

$$\nu = 1/\alpha.$$

5. The Associated Operators

The key to getting around the difficulty raised by the term $c \times k$ mentioned in the last section is to observe that if $\phi(c)$ is a wave function in quantum mechanics, then the Fourier variable k corresponds to momentum and $c \times k$ corresponds to angular momentum. This naturally leads to the consideration of wave function with fixed total angular momentum; i.e., we consider ϕ in $\mathcal{S}$ having the form

$$\phi(c) = Y_\ell(c, z) \psi(|c|) \quad (29)$$

and denote this ϕ as $\phi \in \mathcal{S}^\ell(\ell)$, where $Y_\ell(c, z)$ is an ℓ -th order spherical harmonics of the vector c with z as the polar axis. The Fourier transform $\hat{\phi}(k)$ is also in $\mathcal{S}^\ell(\ell)$:

$$\hat{\phi}(k) = \xi(|k|) Y_\ell(k, z) . \quad (30)$$

We now write the term $H(\cos\theta) \exp(ik \cdot c)$ in (25) as

$$H(\cos\theta) e^{ik \cdot c} = Q \left(\frac{2}{1 + \sin^2 \theta} \right)^{\frac{3}{2}} \exp \left[- \frac{\cos^2 \theta}{2(1 + \sin^2 \theta)} (c^2 + k^2) \right] \quad (31)$$

$$\begin{aligned} Q &= J_0 \left(\frac{\sin^2 \theta}{1 + \sin^2 \theta} |c \times k| \right) \exp \left[\left(1 - \frac{\cos^2 \theta}{1 + \sin^2 \theta} \right) ic \cdot k \right] \\ &= \zeta^{-\frac{1}{2}} \sum_{m=0}^{\infty} a_m J_{m+\frac{1}{2}}(\zeta) P_m(\sin\theta) P_m(\cos \textcircled{H}) \end{aligned} \quad (32)$$

where P_m are the Legendre polynomials, $\zeta = 2|c||k|\sin\theta/(1+\sin^2\theta)$, $\textcircled{H}$ is the angle between c and k , and a_m can be found in Watson [12].

It is noted that in (32) only the term $m = \ell$ can act on (30) while the rest of the terms vanish in the integration over $\textcircled{H}$. We can therefore use a new function Q_ℓ

$$\begin{aligned} Q_\ell &= P_\ell(\sin\theta) \zeta^{-\frac{1}{2}} \Sigma a_m J_{m+\frac{1}{2}}(\zeta) P_m(\cos \textcircled{H}) \\ &= P_\ell(\sin\theta) \exp \left[\frac{2 \sin \theta}{1 + \sin^2 \theta} i c \cdot k \right] \end{aligned} \quad (33)$$

where Q_ℓ has the same result as Q when acting on $\hat{\phi}$ of the form in (30), i.e. on $\phi \in \mathcal{S}(\ell)$.

Now using this Q_ℓ to replace Q in (25) we have a new symbol $q_\ell(c, k)$ instead of $q(c, k)$

$$q_\ell(c, k) = \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin\theta [1 - H_\ell(\cos\theta)] d\theta \quad (34)$$

$$\begin{aligned} H_\ell(x) &= \left[\frac{2}{(2-x^2)} \right]^{\frac{3}{2}} P_\ell(y) \exp \left\{ \frac{-\frac{1}{2}[(c^2+k^2)x^2 + 2(1-y)^2 i c \cdot k]}{(2-x^2)} \right\} \\ y &= (1-x^2)^{\frac{1}{2}} \end{aligned} \quad (35)$$

With a similar procedure as that used in the step from (32) to (33), we can construct a new symbol $r_\ell(c, k)$ in place of $r(c, k)$ as

$$\begin{aligned} r_\ell(c, k) &= \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin\theta \left[(2)^{\frac{3}{2}} \exp \left(-\frac{1}{2}c^2 - \frac{1}{2}k^2 - i k \cdot c \right) - h_\ell(\sin\theta) \right] d\theta \\ h_\ell(\sin \theta) &= H(\sin \theta) \quad \frac{1}{4}\pi \leq \theta \leq \frac{1}{2}\pi, \\ &= H_\ell(\sin \theta) \quad 0 \leq \theta \leq \frac{1}{4}\pi, \end{aligned} \quad (36)$$

where H_ℓ is given in (35), and $H(\sin \theta)$ is given in (27). We introduce

$$L_\ell \phi = \int e^{ik \cdot c} p_\ell(c, k) \hat{\phi}(k) d^3k, \quad (37)$$

$$p_\ell(c, k) = q_\ell(c, k) + r_\ell(c, k). \quad (38)$$

and it is clear from the construction of q_ℓ and r_ℓ that $L\phi = L_\ell\phi$ if $\phi \in \mathcal{S}(\ell)$. We shall call L_ℓ the associated operator of L . If $\phi \notin \mathcal{S}(\ell)$, e.g. $\phi \in \mathcal{S}(m)$ with $m \neq \ell$, then $L\phi \neq L_\ell\phi$.

The main advantage in dealing with the associated operator L_ℓ is that the derivatives of its symbol $p_\ell(c, k)$ behave nicely at infinity. Specifically, we have

$$\operatorname{Re} q_\ell(c, k) > a(c^2 + k^2)^{\frac{1}{4}} + a\ell^{\frac{1}{4}} - a_1 \quad (39)$$

$$|r_\ell(c, k)| < a \quad (40)$$

$$|\partial_c^m \partial_k^n p_\ell(c, k)| < a(1 + k^2)^{-\frac{1}{4}}, \quad |n| + |m| \geq 1 \quad (41)$$

where a and a_1 are positive and independent of ℓ .

We first go to (40). To verify (40), we split the integral in (36) into two parts, one for $0 \leq \theta \leq \frac{1}{4}\pi$ and one for $\frac{1}{4}\pi \leq \theta \leq \frac{1}{2}\pi$. In the interval $\frac{1}{4}\pi \leq \theta \leq \frac{1}{2}\pi$, we can factor out the exponential factor $\exp(-\frac{1}{2}c^2 - \frac{1}{2}k^2 - ik \cdot c)$ which ensures the boundedness of this integral for all c and k ; in fact, it ensures the decay in k in (41) for the integral and its derivatives. In the other interval $0 \leq \theta \leq \frac{1}{4}\pi$, $\beta(\theta, 4)$ is finite and (40) follows from this.

To prove (41), we only note that $q_\ell(x, k)$ and $h_\ell(\sin \theta)$ only contain c^2 , k^2 and $c \cdot k$ and therefore differentiation with respect to c and k can be easily carried out and (41) can be established almost straightforwardly.

To prove (39), we split (34) into three terms

$$q_\ell(c, k) = Q_1 + Q_2 + Q_3, \quad (42)$$

$$Q_1 = \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin \theta [1 - P_\ell(y)] e^{-\Gamma_0} d\theta, \quad (43)$$

$$Q_2 = \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin \theta [1 - e^{-\Gamma_0}] d\theta, \quad (44)$$

$$Q_3 = \int_0^{\frac{1}{2}\pi} \beta(\theta, 4) \sin \theta P_\ell(y) e^{-\Gamma_0} [1 - \Gamma_1] d\theta, \quad (45)$$

where the functions Γ_0 and Γ_1 are

$$\Gamma_0 = \frac{1}{2} x^2 (c^2 + k^2) / (2 - x^2), \quad (46)$$

$$\Gamma_1 = [2 / (2 - x^2)]^{3/2} \exp[-i(1 - y)^2 c \cdot k / (2 - x^2)]. \quad (47)$$

We first note that Q_3 is bounded in all c , k and ℓ .

For Q_2 , we can introduce a new integration variable.

$$t = (c^2 + k^2)^{1/2} x \quad (48)$$

and show that

$$\operatorname{Re} Q_2 > 2a(c^2 + k^2)^{1/4} - \frac{1}{2} a_1 \quad (49)$$

for some positive a and a_1 .

On the other hand, for Q_1 we can use the formula of Hilb; see Szegő [9]:

$$P_\ell(\sin \theta) = J_0[(\ell + \frac{1}{2})(\frac{1}{2}\pi - \theta)] + R_\ell(\theta), \quad (50)$$

$$|R_\ell(\theta)| \leq a|\theta|. \quad (51)$$

Now we substitute (50) into (43), introduce a new integration variable

$$t = (\ell + \frac{1}{2})(\frac{1}{2}\pi - \theta),$$

and use the inequality

$$\Gamma_0 \leq \frac{1}{2} t^2 (c^2 + k^2) / (\ell + \frac{1}{2})^2.$$

As a result, we obtain for Q_1

$$\begin{aligned} \operatorname{Re} Q_1 &\geq \beta_0 \ell^{\frac{1}{2}} \int_0^1 t^{-1-2\nu} [1 - J_0(t)] \exp[-\frac{1}{2}(c^2 + k^2) / (\ell + \frac{1}{2})^2] dt \\ &\quad + b(c, k) \end{aligned}$$

where $b(c, k)$ is uniformly bounded for all c, k and ℓ . In the above estimate we have used the explicit form of the singularity of $\beta(\theta, x)$ at $\theta = \frac{1}{2}\pi$ given at the end of Sec. 4. An alternative way of writing the last estimate is

$$\operatorname{Re} Q_1 \geq a \ell^{\frac{1}{2}} - a_1, \quad c^2 + k^2 < \ell^2 \quad (52)$$

while for $c^2 + k^2 > \ell^2$ we can use (49) to get a $\ell^{1/2}$ factor.

In any case, (49) and (52) together with the boundedness of Q_3 lead to the desired result (39).

6. The Eigenvalue Problem

We now consider the eigenvalue problem

$$\lambda\phi = A_\ell\phi = (N + L_\ell)\phi \quad (53)$$

where L_ℓ is given in (37) and N is a large but fixed positive number. What we like to do is to invert the operator A_ℓ . In doing this, we essentially follow the general procedure described in Kumano-go [7], making use of the four theorems in our Sec. 3.

First we rewrite the operator A_ℓ as a PsDO with multiple symbol in the form

$$A_\ell\phi = \int \int e^{ik \cdot (c-\eta)} a_\ell(c,k) \phi(\eta) d^3\eta d^3k, \quad (54)$$

$$a_\ell(c,k) = N + p_\ell(c,k) \quad . \quad (55)$$

The symbol $a_\ell(c,k)$ may be expressed as

$$a_\ell(c,k) = a_\ell(\eta,k) + b(c,\eta,k) \quad (56)$$

$$b(c,\eta,k) = (c-\eta) \cdot \int_0^1 [\partial_c p_\ell(\eta + x(c-\eta), k)] dx \quad (57)$$

Now substituting (56) and (57) into (54) and integrating by parts yield

$$A_\ell\phi = A_\ell^R\phi + T\phi \quad (58)$$

where A_ℓ^R is the reverse operator [7] of A_ℓ and A_ℓ^R and T are given by

$$A_{\ell}^R \phi = \int \int e^{ik \cdot (c-\eta)} a_{\ell}(\eta, k) \phi(\eta) d\eta dk \quad (59)$$

$$T\phi = \int \int e^{ik \cdot (c-\eta)} t(c, \eta, k) \phi(\eta) d^3\eta d^3k \quad (60)$$

$$t(c, \eta, k) = i \int_0^1 [\partial_k \cdot \partial_c a_{\ell}(\eta + x(c-\eta), k)] dx \quad (61)$$

It is noted that in the integration by parts we have to follow the procedure of Kumano-go (see Ref. 7, pp. 421-422), using the fact the $\phi \in \mathcal{S}$. According to (41), the multiple symbol $t(c, \eta, k)$ satisfies the condition in Theorem 4 and consequently T , which maps $\mathcal{S}$ into $\mathcal{L}^2$, can be extended to a bounded operator from $\mathcal{L}^2$ to $\mathcal{L}^2$ with $\|T\|$ independent of N .

Next we introduce the operator B defined by

$$B\phi = \int e^{ik \cdot c} [a_{\ell}(c, k)]^{-1} \hat{\phi}(k) d^3k \quad (62)$$

where $\operatorname{Re} a_{\ell}(c, k) > \frac{N}{8}$, if N is large enough, according to (39) and (40). Now Theorems 1 and 2 imply that B is a compact operator on $\mathcal{L}^2$ and

$$\|B\| < b/N \quad (63)$$

for some constant b .

We then consider the composite operator

$$BA_{\ell} = BA_{\ell}^R \phi + BT\phi \quad (64)$$

and write the operator BA_{ℓ}^R as

$$BA_{\ell}^R \phi = \int \int e^{ik \cdot (c-\eta)} [a_{\ell}(\eta, k)/a_{\ell}(c, k)] \phi(\eta) d^3\eta d^3k \quad (65)$$

Now substituting (56) for $a_\ell(\eta, k)$ in (65), we obtain

$$BA_\ell^R \phi = \phi(c) + Y\phi \quad (66)$$

$$Y\phi = \int \int e^{ik(c-\eta)} y(c, \eta, k) \phi(\eta) d^3\eta d^3k \quad (67)$$

$$y(c, \eta, k) = i \int_0^1 dx \partial_k \cdot [a_\ell^{-1}(c, k) \partial_c a_\ell(\eta + x(c-\eta), k)] \quad (68)$$

From (41), we see that $y(c, \eta, k)$ also satisfies the condition in Theorem 4 and

$$\|Y\| < b/N \quad (69)$$

for some constant b , independent of ℓ .

Now summarizing the result, we have

$$BA_\ell \phi = (1 + Y + BT)\phi \quad (70)$$

and the bounds in (63) and (69) show that

$$\|Y + BT\| < b/N \quad (71)$$

for some b , from which it is concluded that if N is large enough then $(BA_\ell)^{-1}$ exists and is bounded in $\mathcal{L}^2$. Therefore the eigenvalue problem (53) may be written as

$$\frac{1}{\lambda} \phi = (BA_\ell)^{-1} B\phi = A_\ell^{-1} \phi \quad (72)$$

and since B is compact from $\mathcal{L}^2$ to $\mathcal{L}^2$ and $(BA_\ell)^{-1}$ is bounded it follows that A_ℓ^{-1} is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$.

Next, we wish to show that as $\ell \rightarrow \infty$, we have $\|A_\ell^{-1}\| \rightarrow 0$. In this connection, we first note that because of (39) we have

$$|[a_\ell(c,k)]^{-1}| \rightarrow 0$$

as $\ell \rightarrow \infty$. This, by Theorem 1 of Sec. 3, shows that $\|B\| \rightarrow 0$ as $\ell \rightarrow \infty$. On the other hand, the constant b in (71) can be made independent of ℓ , and therefore

$$\|A_\ell^{-1}\| \rightarrow 0 \quad (72)$$

as $\ell \rightarrow \infty$.

The fact that A_ℓ^{-1} is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$ together with the result in (72) leads to the following conclusions.

- (1) A^{-1} is a compact operator from $\mathcal{X}^2$ to $\mathcal{L}^2$.
- (2) The eigenfunction of A are all $\mathcal{L}^2$ -functions. Since $A = N+L$, the eigenfunction of L are all $\mathcal{L}^2$ -functions.
- (3) The spectrum of L is purely discrete.
- (4) The $\mathcal{L}^2$ -eigenfunctions of L are complete in $\mathcal{L}^2$, since L is symmetric and non-negative.
- (5) Further computation shows that the eigenfunctions of L in fact belong to $\mathcal{S}$ where $\mathcal{S}$ is the set of C^∞ -functions which, together with their derivatives, decays at infinity faster than any inverse powers of c . This will be proved in Part II for the general case.

(6) An alternative way of stating the first conclusion is that L^{-1} is a compact operator from $\mathcal{L}_1^2$ to $\mathcal{L}_1^2$.

7. Remarks

Our analysis relies on two main tools. One is the theory of pseudo-differential operators and the other the use of the associated operator A_ℓ . In constructing A_ℓ , the major step is the step from (32) to (33). Since only the term with $m = \ell$ in (32) need be kept unchanged, there is considerable freedom in altering the other terms with $m \neq \ell$. One way would be to set all of them to zero, leading to the result

$$Q_\ell = \zeta^{-\frac{1}{2}} A_\ell J_{\ell+\frac{1}{2}}(\zeta) P_\ell(\sin\theta) P_\ell(\cos\theta) .$$

If we do this, we would end up with a one-dimensional equation with $|c|$ as the independent variable, which is what Wang Chang and Uhlenbeck have in Ref. 11. Then we either have to construct the eigenfunctions or have to deal with (instead of three-dimensional Fourier transforms) one-dimensional Hankel transforms for which there exists no pseudo-differential operator theory. This consideration leads us to the construction in (33) which retains the pseudo-differential operator formalism.

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Part II

General Repulsive Inverse-Power Potentials

1. Introduction

This paper studies the eigenvalue problem of the collision operator in the linearized Boltzmann equation with a repulsive inverse-power intermolecular potential of the form

$$V(r) = ar^{-\alpha}, \quad \alpha > 2 \quad (1)$$

where r is the intermolecular distance.

The purpose of this paper is to show that the collision operator L [see eqn. (12)] has a purely discrete spectrum and its eigenfunctions exist, are $\mathcal{L}^2$ -functions and form a complete orthonormal set in $\mathcal{L}^2$. The result is obtained by showing that L^{-1} is a compact operator from $\mathcal{L}_\perp^2$ to $\mathcal{L}_\perp^2$, where $\mathcal{L}_\perp^2$ is the subspace of $\mathcal{L}^2$ orthogonal to the five eigenfunctions of L for the eigenvalue zero, and $\mathcal{L}^2$ are the square integrable functions.

The method we use here follows closely the procedure carried out in the first part of this study [4] (hereafter referred to as Part I), for the special case of $\alpha = 4$, i.e. the Maxwell's gas. It is suggested that Part I be read as a preliminary since the present report makes frequent use of the results given in Part I.

In the next section we consider the operator $A = N+L$, where N is a large positive number and L is the symmetrized collision operator in (12), and we construct the associated operator A_ℓ in Sec. 3. We use the formalism of pseudo-differential operators, [see Sec. 3, Part I]. The properties

of the symbol of the operator A_ℓ is investigated in Sec. 4. These properties (in terms of bounds for the symbol of A_ℓ and its derivatives) are then used in Sec. 5 to show that A^{-1} exists and is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$. From this we conclude that the spectrum of L is purely discrete and its eigenfunctions are $\mathcal{L}^2$ -functions and they are complete in $\mathcal{L}^2$. Further computations show that the eigenfunctions of L are not only $\mathcal{L}^2$ but also belong to $\mathcal{S}$, the rapidly decreasing C^∞ -functions. These computations are sketched in Sec. 6, followed by some general remarks in Sec. 7.

2. Collision Operators

Consider the collision operator C in the linearized Boltzmann equation

$$C[f] = \int K'(c, \eta) [f(\eta) - f(c)] \exp(-\eta^2) d^3\eta \\ + \int [K'_1(c, \eta) - K_1(c, \eta)] f(\eta) \exp(-\eta^2) d^3\eta \quad (2)$$

where kernels K' , K'_1 and K_1 correspond, respectively, to terms involving what is conventionally denoted as f' , f'_1 and f_1 in the collision integral. These kernels may be written as [1]

$$K'(c, \eta) = \int \sin\theta G(\tan\theta) \sec^3\theta \sigma(|g| \sec\theta, \theta) d\theta \quad (3)$$

$$K'_1(c, \eta) = \int \sin\theta G(\cot\theta) \csc^3\theta \sigma(|g| \csc\theta, \theta) d\theta \quad (4)$$

$$K_1(c, \eta) = \int \sin\theta \sigma(|g|, \theta) d\theta \quad (5)$$

$$g = \eta - c \quad (6)$$

$$G(y) = I_0(2|c \wedge g|y) \exp(-y^2 g^2) \quad (7)$$

where we have used c , η , g to denote vectors and $|g|$, $|c|$, $|\eta|$ to denote scalars and the θ -integration in (3)-(5) is from 0 to $\frac{1}{2}\pi$. The function $\sigma(|g|, \theta)$ is related to the differential cross-section and for inverse-power potentials given in (1),

$$\sigma(|g|, \theta) = |g|^\gamma \beta(\theta, \alpha) \quad (8)$$

where $\gamma = 1 - 4\nu$, $\nu = 1/\alpha$, $\gamma > -1$ for $\alpha > 2$, and $\beta(\theta, \alpha)$

diverges as $(\frac{1}{2}\pi - \theta)^{-1-2\nu}$ as $\theta \rightarrow \frac{1}{2}\pi$. This property of $\beta(\theta, \alpha)$ may be expressed as

$$\beta(\theta, \alpha) = \beta_0 (\cos \theta)^{1-4\nu} (\tan \theta)^{2-2\nu} + B(\tan \theta) \quad (9)$$

$$|B(\tan \theta)| < b(1 + \tan^2 \theta)^\nu \quad (10)$$

which will be found useful later. We may just set $\beta_0 = 1$ on account of the linearity of the eigenvalue problem. Throughout this paper we shall use b to denote positive constants, not all of them necessarily the same.

It is expedient to use

$$\phi(c) = f(c) \exp(-\frac{1}{2}c^2) \quad (11)$$

and consider the operator

$$L\phi = -\exp(-\frac{1}{2}c^2) C[\phi(c) \exp(\frac{1}{2}c^2)] . \quad (12)$$

which can be shown to map $\mathcal{S}$ into $\mathcal{L}^2$ (this was discussed in sec. 3 of Part I) and is symmetric and non-negative. Consequently L can be considered as a non-negative self-adjoint (but unbounded) operator in $\mathcal{L}^2$. Our main task is the study of the eigenvalue problem

$$\lambda \phi = (N+L)\phi = A\phi \quad (13)$$

where N is a large but fixed positive number.

To write A in a pseudo-differential operator (PsDO) form, it suffices to use the Fourier transform

$$\hat{\phi}(k) = (2\pi)^{-3} \int e^{-ik \cdot c} \phi(c) d^3c \quad (14)$$

in terms of which $L\phi$ becomes

$$L\phi = \int e^{ik \cdot c} p(c, k) \hat{\phi}(k) d^3k \quad (15)$$

$$p(c, k) = q(c, k) + r(c, k) + u(c) \quad (16)$$

$$q(c, k) = \int d^3g K'(c, \eta) [1 - e^{ik \cdot g}] \exp(-\frac{1}{2}c^2 - \frac{1}{2}\eta^2) \quad (17)$$

$$r(c, k) = \int d^3g [K_1(c, \eta) - K_1'(c, \eta)] \exp(-\frac{1}{2}c^2 - \frac{1}{2}\eta^2 + ik \cdot g) \quad (18)$$

$$u(c) = \int d^3g K'(c, \eta) [\exp(-\frac{1}{2}\eta^2) - \exp(-\frac{1}{2}c^2)] \exp(-\frac{1}{2}\eta^2) . \quad (19)$$

Equation (15) gives $L\phi$ in the form of a PsDO and $p(c, k)$ is the symbol of the PsDO L . Using the analyses in Sec. 3, we can show that $p(c, k)$ satisfies condition (12) in Part I for $\sigma = \mu = \nu$, and consequently L maps $\mathcal{D}$ into $\mathcal{D}'$, according to Theorem 1, Sec. 3, of Part I. But we shall not carry out the detailed estimates since we shall not have to use the estimates of $p(c, k)$ and its derivatives in our subsequent calculations.

The operator A then becomes

$$A\phi = \int e^{ik \cdot c} a(c, k) \hat{\phi}(k) d^3k , \quad (20)$$

$$a(c, k) = N + p(c, k) . \quad (21)$$

3. The Associated Operators

As it stands, the operator A in (19) is difficult to invert. This difficulty has been discussed in the special case of $\alpha = 4$ (the case of Maxwell molecules) and it can only become worse for the general case of $\alpha \neq 4$. The way out of this is, as we have done in the case of Maxwell molecules, to construct the associated operator A_ℓ (see Sec. 5, Part I). The construction for such an A_ℓ in the case of general inverse-power potential with $\alpha \neq 4$, unlike the case of Maxwell molecules, is somewhat complicated and a brief explanation would be quite helpful. This may be given in the following heuristic arguments.

The main difficulty is in the symbol $q(c,k)$ which arises from the term

$$C_o[f] = \int d^3n \exp(-n^2) \int d\phi \int d\theta |g|^\gamma \beta(\theta, \alpha) [f(c') - f(c)]$$

in $C[f]$. However, we observe that for large c

$$C_o[f] \sim |c|^\gamma \int d^3n \exp(-n^2) \int d\phi \int d\theta \beta(\theta, \alpha) [f(c') - f(c)] \quad (22)$$

which has the same form as the collision operator for the Maxwell molecules except for the factor $\beta(\theta, \alpha)$ instead of $\beta(\theta, 4)$ and the extra $|c|^\gamma$, both of which can be easily incorporated into the treatment in Part I. It is therefore natural that we shall explore this large- c behavior. This will now be done in a rigorous manner in the following.

We shall now construct the associated operator A_ℓ in four steps.

First we rewrite $q(c,k)$ as

$$q(c,k) = s(c,k) + v(c,k) \quad (23)$$

$$v(c,k) = q(c,k) - s(c,k) \quad (24)$$

$$s(c,k) = |c|^\gamma \int d^3\eta S(c,\eta) [1 - e^{ik \cdot \eta}] \exp(-\frac{1}{2}c^2 - \frac{1}{2}\eta^2) \quad (25)$$

$$S(c,\eta) = \int_0^{\frac{1}{2}\pi} \sin\theta G(\tan\theta) \beta(\theta,\alpha) \sec^3\theta (\sin\theta)^{-1+4\nu} d\theta \quad (26)$$

where $G(\tan\theta)$ is given in (7) and we note that the kernel $S(c,\eta)$ is precisely the kernel for (22) except for the extra factor $(\sin\theta)^{4\nu-1}$. Since the singularity of $\beta(\theta,\alpha)$ is at $\theta = \frac{1}{2}\pi$, the extra factor, being equal to 1 at $\theta = \frac{1}{2}\pi$, does not have any particular significance and is added here merely for later computational convenience.

The next step is to introduce a C^∞ -function $\zeta(c)$ with the property that for some $c_0 > 1$, $\zeta(c) \equiv 1$ for $|c| < c_0$, $\zeta(c) \equiv 0$ for $|c| > 2c_0$, and $0 \leq \zeta(c) \leq 1$ for $c_0 < |c| < 2c_0$, and we write

$$q(c,k) = \zeta(c)q(c,k) + [1-\zeta(c)][s(c,k)+v(c,k)] \quad (27)$$

So far, nothing has changed in $q(c,k)$; we merely rewrite $q(c,k)$ in a form from which it is convenient to construct the new operator A_λ .

The third step is to rewrite $s(c,k)$ in (25) as

$$s(c,k) = |c|^\gamma \int d^3\eta \int d\phi d\theta \beta^*(\theta) [e^{ik \cdot c} - e^{ik \cdot c'}] \times \exp[-\frac{1}{2}c^2 - \eta^2 - ik \cdot c + \frac{1}{2}(c')^2] \quad (28)$$

which is just the type of integral that appears in the collision integral for the Maxwell molecules. Here $\beta^*(\theta)$ is

$$\beta^*(\theta) = \beta(\theta, \alpha)(\sin\theta)^{4\nu-1}. \quad (29)$$

The integrals in (28) can be evaluated by using the formula of Waldmann [5] in conjunction with analytic continuation as we have shown in Sec. 5, Part I. The result becomes

$$\begin{aligned} s(c, k) = & |c|^\gamma \int_0^{\frac{1}{2}\pi} \beta^*(\theta) (2t)^{3/2} \\ & \times \exp(-\frac{1}{2}t \cos^2 \theta c^2) \left[1 - J_0(t \sin 2\theta |c \times k|) \right. \\ & \left. \times \exp\{-\frac{1}{2}t \cos^2 \theta (k^2 + 2ic \cdot k)\} \right] d\theta \end{aligned} \quad (30)$$

$$t = (1 + \sin^2 \theta)^{-1} \quad (31)$$

We can now repeat the procedure in Sec. 5 of Part I to construct a new symbol. In fact, we consider

$$\int e^{ik \cdot c} s(c, k) \hat{\phi}(k) d^3 k \quad (32)$$

for $\phi \in \mathcal{S}(\mathbb{R}^3)$, viz.

$$\phi(c) = Y_\ell(c, z) \psi(|c|) \quad (33)$$

where $Y_\ell(c, z)$ is the ℓ -th order spherical harmonics for c with polar axis z . Then $\hat{\phi}(k)$ is also in $\mathcal{S}(\mathbb{R}^3)$, and we follow the same step of Part I [see eqns. (32) and (33) of Part I] and obtain a new symbol

$$s_{\ell}(c,k) = |c|^{\gamma} \int_0^{\frac{1}{2}\pi} \beta(\theta, \alpha) (\sin\theta)^{4\nu} (2t)^{3/2} \exp(-\frac{1}{2}t \cos^2\theta c^2) \\ \times [1 - P_{\ell}(\sin\theta) \cdot \exp(-\Gamma)] \quad (34)$$

$$\Gamma = \frac{1}{2}t[k^2 \cos^2\theta + 2(1-\sin\theta)^2 i c \cdot k] \quad (35)$$

where $t = \frac{1}{(1+\sin^2\theta)}$. We note that for $\phi \in \mathcal{S}'(\mathbb{R})$ in (33),

we have

$$\int e^{ik \cdot c} s(c,k) \hat{\phi}(k) d^3k = \int e^{ik \cdot c} s_{\ell}(c,k) \hat{\phi}(k) d^3k. \quad (36)$$

For a detailed discussion of this point we refer to Sec. 5 of Part I.

Now the final step is to construct a new symbol $q_{\ell}(c,k)$ from (27) as

$$q_{\ell}(c,k) = \varepsilon \zeta(c) s_{\ell}(c,k) + [1-\zeta(c)][s_{\ell}(c,k) + v(c,k)] \\ + \zeta(c)[q(c,k) - \varepsilon s(c,k)] \quad (37)$$

where we have changed two $s(c,k)$ terms to $s_{\ell}(c,k)$ with others remaining unchanged as in (27).

With this new symbol $q_{\ell}(c,k)$, two new operators L_{ℓ} and A_{ℓ} may be introduced as

$$p_{\ell}(c,k) = q_{\ell}(c,k) + r(c,k) + u(c) \quad (38)$$

$$L_{\ell} \phi = \int e^{ik \cdot c} p_{\ell}(c,k) \hat{\phi}(k) dk, \quad (39)$$

$$A_{\ell} \phi = \int e^{ik \cdot c} a_{\ell}(c,k) \hat{\phi}(k) dk, \quad (40)$$

$$a_{\ell}(c,k) = N + p_{\ell}(c,k) \quad (41)$$

where the main point to note is that $A\phi = A_\ell\phi$ for $\phi \in \mathcal{D}(\ell)$ in (33). It should be pointed out that since L is isotropic all eigenfunctions of A are in $\mathcal{D}(\ell)$ for some ℓ .

What we shall show is that A_ℓ^{-1} exists and is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$. This we shall prove by using the theorems of PsDO given in Sec. 3, Part I. For this purpose some properties of the symbols $q_\ell(c,k)$, $r(c,k)$ and $u(c)$ have to be established; this is undertaken in the next section. These estimates can also be used to show that $\|A_\ell^{-1}\| \rightarrow 0$ as $\ell \rightarrow \infty$, which leads to the fact that A^{-1} is compact from $\mathcal{L}^2$ to $\mathcal{L}^2$.

4. Properties of the Symbols

The purpose of this section is to establish some bounds for the various symbols and their derivatives with respect to c and k . These estimates involve considerable computations. But they are of critical importance to our theory and some of them are not at all very obvious. For these reasons we shall present here the major steps in the proofs of these estimates. It is recalled that throughout this paper we shall use b to denote positive constants, not necessarily all of them the same constant.

Let us first summarize the main results of this section:

$$N + \operatorname{Re} a_\ell(c, k) > \frac{1}{8}N + b(1+c^2)^{\frac{1}{2}\gamma}(c^2+k^2)^\nu + b\ell^{2\nu} \quad (42)$$

$$|\partial_c^m a_\ell(c, k)| < b(1+k^2)^\nu, \quad |m| \geq 1 \quad (43)$$

$$|\partial_c^m \partial_k^n a_\ell(c, k)| < b(1+k^2)^{-\nu} 1, \quad |n| \geq 1 \text{ and } |m| \geq 0 \quad (44)$$

where the constants b in (42) - (44) are independent of ℓ

and $\nu_1 = \nu(1-2\nu)/(1+2\nu) > 0$ and Re denotes the real part

and we have used the abbreviation

$$\partial_c^m = \partial_{c_1}^{m_1} \partial_{c_2}^{m_2} \partial_{c_3}^{m_3}, \quad (45)$$

$$|m| = m_1 + m_2 + m_3, \quad (46)$$

where the c_i , $i=1,2,3$ are the three components of the vector c . Similar notation is also used for ∂_k^n .

The results in (42), (43) and (44) are all that are

needed for the analyses in the next section. We now embark on the proof for these estimates by considering the terms $u(c)$, $r(c,k)$, $\zeta(c)q(c,k)$, $v(c,k)$, $s_\rho(c,k)$ and $\zeta(c)s(c,k)$ in (17), (18), (19), (24), (34) and (30), one at a time. Those not interested in the details of the proofs can skip the rest of this section.

(A) Estimates for $u(c)$

To obtain the desired estimates for $u(c)$, we shall reduce the five-fold integral in (19) into a one-fold integral of confluent hypergeometric functions. To this end, we denote ψ as the angle between c and g and consider

$$U = \int \sin\psi d\psi \cdot I_0(2|c \times g| \tan \theta) [\exp(-\eta^2) - \exp(-\frac{1}{2}\eta^2 - \frac{1}{2}c^2)]$$

which can be integrated out in terms of Bessel functions of order $\frac{1}{2}$ as

$$U = (2\pi)^{\frac{1}{2}} [I_{\frac{1}{2}}(\zeta_1) \zeta_1^{-\frac{1}{2}} - I_{\frac{1}{2}}(\zeta_2) \zeta_2^{-\frac{1}{2}}] \quad (47)$$

$$\zeta_1 = 2|c||g|/\cos\theta \quad (48)$$

$$\zeta_2 = 2|c||g|(\tan^2\theta + \frac{1}{4})^{\frac{1}{2}} \quad (49)$$

by using the well-known addition theorem of Bessel functions, see eqn. (9) on page 370 in Watson [6].

Now substituting (47) into (19) and using the table for Hankel transforms [3] yields

$$u(c) = b \int_0^{\frac{1}{2}\pi} d\theta \beta(\theta, \alpha) (\sec\theta)^{3+\gamma} \sin\theta \, W \exp(-c^2) \quad (50)$$

where b is a positive constant whose exact value is of no importance to our analysis, and W is

$$W = (\cos \theta)^{3+\gamma} {}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, c^2\right) - \Omega(\theta) {}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, c^2 - \delta c^2\right), \quad (51)$$

$$\Omega(\theta) = \left(\tan^2 \theta + \frac{1}{4}\right)^{\frac{1}{4}} \left(\tan^2 \theta + \frac{1}{2}\right)^{\frac{1}{4} - 2 - \frac{1}{2}\gamma}, \quad (52)$$

$$\delta = \frac{1}{2} \left(\tan^2 \theta + \frac{1}{2}\right)^{-1}, \quad (53)$$

where ${}_1F_1$ is the confluent hypergeometric function.

To obtain estimates for $u(c)$ in (50) and its derivatives, it is convenient to regroup W in (51) as

$$W = W_1 + W_2 \quad (54)$$

$$W_1 = [(\cos \theta)^{3+\gamma} - \Omega(\theta)] {}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, c^2\right) \quad (55)$$

$$W_2 = \Omega(\theta) \left[{}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, c^2\right) - {}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, c^2 - \delta c^2\right) \right] \quad (56)$$

Substituting (54) into (50) and integrating by parts with θ for the W_2 term yields

$$u(c) = u_1(c) + u_2(c) \quad (57)$$

$$u_1(c) = a_0 {}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, c^2\right) \exp(-c^2) \quad (58)$$

$$u_2(c) = bc^2 \int_0^{\frac{1}{2}\pi} d\theta \frac{d\delta}{d\theta} \tilde{\beta}(\theta) {}_1F_1\left(\frac{5}{2} + \frac{1}{2}\gamma, \frac{5}{2}, c^2 - \delta c^2\right) \exp(-c^2) \quad (59)$$

where a_0 is the integral in (50) of the bracketed term in (55) which may be negative, and $\tilde{\beta}(\theta)$ is

$$\tilde{\beta}(\theta) = (1 + \gamma/3) \int_0^\theta d\theta \sin\theta (\sec\theta)^{3+\gamma} \beta(\theta, \alpha) \Omega(\theta) . \quad (60)$$

The behavior of $\tilde{\beta}(\theta)$ can be obtained from (52), (9) and (10) and may be expressed as

$$\tilde{\beta}(\theta) = \frac{-1}{1+2\nu} \delta^{-\nu} + o(\delta^{-\nu+\frac{1}{2}}) \quad (61)$$

where δ is given in (53).

For large $|c|$, the confluent hypergeometric function has the asymptotic behavior [2]

$${}_1F_1\left(\frac{5}{2} + \frac{1}{2}\gamma, \frac{5}{2}, c^2 - \delta c^2\right) \approx \left[\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2} + \frac{1}{2}\gamma)} \right] [c^2(1-\delta)]^{\frac{\gamma}{2}} \exp(-\delta c^2 + c^2) \quad (62)$$

Substituting this into (59), we have for large c

$$u_2(c) \approx \frac{b}{1+2\nu} |c|^{\gamma+2\nu} \left[\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2} + \frac{1}{2}\gamma)} \right] \int_0^\infty e^{-x} x^\nu dx \quad (63)$$

and on the other hand, from (58) we have for large c

$$|u_1(c)| \approx b |c|^\gamma . \quad (64)$$

With the result in (63) and (64), the following estimates are obtained

$$u(c) > b |c|^{\gamma+2\nu} - \frac{1}{8} N \quad (65)$$

$$|u(c)| < b(1+c^2)^{\frac{1}{2}\gamma+\nu} \quad (66)$$

for all c if N is taken large enough.

The estimates for the derivatives of u_1 and u_2 can easily be obtained as follows. First we use [6]

$${}_1F_1\left(\frac{3}{2} + \frac{1}{2}\gamma, \frac{3}{2}, +c^2\right) \exp(-c^2) = {}_1F_1\left(-\frac{1}{2}\gamma, \frac{3}{2}, -c^2\right) \quad (67)$$

$$\frac{\partial}{\partial z} {}_1F_1\left(-\frac{1}{2}\gamma, \frac{3}{2}, z\right) = {}_1F_1\left(1 - \frac{1}{2}\gamma, \frac{5}{2}, z\right) \left(-\frac{\gamma}{3}\right) \quad (68)$$

$$|{}_1F_1\left(1 - \frac{1}{2}\gamma, \frac{5}{2}, z\right)| \leq b|z|^{\frac{\gamma}{2}-1} \text{ as } z \rightarrow -\infty, \quad (69)$$

which shows that each differentiation with respect to c brings down a power of $|c|$ as $c \rightarrow \infty$. Similar argument applies to $u_2(c)$, and we therefore have

$$|\partial_c^m u(c)| < b \quad |m| \geq 1 \quad (70)$$

The estimates in (65), (66) and (70) are the desired results.

(B) Estimates for $r(c, k)$

To obtain estimates for $r(c, k)$ in (18) and its derivatives with respect to c and k , we first split the θ -integration into two ranges, $0 \leq \theta \leq \theta_0$, and $\theta_0 < \theta \leq \frac{1}{2}\pi$, $\theta_0 < \frac{1}{2}\pi$, such that $|\cot\theta| < \frac{1}{4}$ in $\theta_0 \leq \theta \leq \frac{1}{2}\pi$. We then write

$$\begin{aligned} r(c, k) &= r_1(c, k) + r_2(c, k) + r_3(c, k) \\ r_1(c, k) &= \int d^3g \int_{\theta_0}^{\pi} d\theta \sin\theta \beta(\theta, \alpha) |g|^\gamma [1 - (\csc\theta)^{3+\gamma} G(\cot\theta)] \\ &\quad \times \exp\left(-\frac{1}{2}c^2 - \frac{1}{2}\eta^2 + ik \cdot g\right) \end{aligned} \quad (71)$$

$$r_2(c,k) = \int d^3g \int_0^{\theta_0} d\theta \beta(\theta, \alpha) (\csc \theta)^{2+\gamma} G(\cot \theta) |g|^\gamma \times \exp(-\frac{1}{2} c^2 - \frac{1}{2} \eta^2 + ik \cdot g) \quad (72)$$

$$r_3(c,k) = \int d^3g \int_0^{\theta_0} d\theta \sin \theta \beta(\theta, \alpha) \times \exp(-\frac{1}{2} c^2 - \frac{1}{2} \eta^2 + ik \cdot g) |g|^\gamma \quad (73)$$

In (73) we note that the function $\beta(\theta, \alpha) \sin \theta$ is absolutely integrable in $0 \leq \theta \leq \theta_0$, and consequently

$$|r_3(c,k)| \leq b \exp(-\frac{1}{2} c^2) \int d\eta \exp(-\eta^2) |c-\eta|^\gamma \leq b(1+c^2)^{\gamma/2} \exp(-\frac{1}{2} c^2) \quad (74)$$

On the other hand, we can write

$$i e^{ik \cdot g} = \frac{1}{|k|} \frac{\partial}{\partial g_{||}} (e^{i|k|g_{||}} - 1) \quad (75)$$

and integrating by parts with respect to $g_{||}$, the component of g parallel to k . We then obtain

$$|r_3(c,k)| \leq b(1+c^2)^{1/2} (1+k^2)^{-1/2} \exp(-\frac{1}{2} c^2) \quad (76)$$

The derivatives of $r_3(c,k)$ with respect to c and k can be estimated similarly and the result may be written as

$$|\partial_c^m \partial_k^n r_3(c,k)| \leq b(1+k^2)^{-1/2} (1+c^2)^{-1/2} \quad (77)$$

For $r_1(c,k)$ and $r_2(c,k)$, we have to use

$$|\partial_c^m \partial_g^n I_0(2|c \times g| \cot \theta)| \leq b(1+c^2)^{|m|+|n|} I_0(2|c \times g| \cot \theta) \quad (78)$$

obtained from the series for I_0 by termwise differentiation.

Now for $r_1(c,k)$, we note that in the range $\theta_0 \leq \theta \leq \frac{1}{2}\pi$, $|\cot \theta| \leq \frac{1}{4}$, and consequently

$$\begin{aligned} |\partial_c^m \partial_g^n G(\cot \theta)| &\leq b(1+c^2)^M I_0(2|c|g|\frac{1}{4}|) \\ &\leq b(1+c^2)^M \exp(\frac{1}{2}|c||\eta|) . \end{aligned} \quad (79)$$

We first integrate (71) by parts with respect to θ , using

$$\tilde{\beta}(\theta) = \int_{\theta_0}^{\theta} d\theta \sin \theta \beta(\theta, \alpha) , \quad (80)$$

we obtain the estimate

$$\begin{aligned} |r_1(c,k)| &\leq b \int d^3g^\gamma (1+g^2)^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} |\tilde{\beta}(\theta)| \\ &\quad \times \exp(-\frac{1}{2}c^2 - \frac{1}{2}\eta^2 + \frac{1}{2}|c||\eta|) d\theta \\ &\leq b[1+c^2]^{1+\frac{\gamma}{2}} \exp(-\frac{1}{4}c^2) . \end{aligned} \quad (81)$$

Now, using the same procedure of partially integrating with respect to $g_{||}$ as in the estimate for $r_3(c,k)$, we have

$$\begin{aligned} |r_1(c,k)| &\leq b[1+c^2]^{2+\frac{\gamma}{2}} \exp(-\frac{1}{4}c^2) \cdot (1+k^2)^{-\frac{1}{2}} \\ &\leq b(1+k^2)^{-\frac{1}{2}} (1+c^2)^{-\frac{1}{2}} \end{aligned} \quad (82)$$

And the derivatives of $r_1(c,k)$ can be estimated in the same fashion by using (79) and the result is

$$|\partial_c^m \partial_k^n r_1(c,k)| \leq b(1+k^2)^{-\frac{1}{2}} (1+c^2)^{-\frac{1}{2}} \quad (83)$$

for $|m| \geq 0$ and $|n| \geq 0$.

Finally we turn to $r_2(c,k)$ which is the most difficult part of $r(c,k)$. We first introduce $x = |g|\cot\theta$ and note that

$$\beta(\theta,\alpha)(\csc\theta)^{2+\gamma}d\theta = \tilde{\beta}(x,|g|)dx, \quad (84)$$

$$|\tilde{\beta}(x,|g|)| \leq \frac{[1+(x/|g|)^\gamma]}{|g|}. \quad (85)$$

which is readily seen from the fact that $\beta(\theta,\alpha)$ is bounded for $0 \leq \theta \leq \theta_0$ with $\theta_0 < \frac{1}{2}\pi$.

We now write (72) as

$$|r_2(c,k)| \leq \int d^3g \int_0^\infty dx |g|^{-1} [|g|^\gamma + x^\gamma] \\ \times I_0(2|c x \sin(\hat{H})|) \exp(-\frac{1}{2}c^2 - \frac{1}{2}x^2 - x^2), \quad (86)$$

where $(\hat{H})$ is the angle between c and g .

It is useful to make the following observations:

(1) The contribution to (86) from the part $|\sin(\hat{H})| < \frac{1}{4}$ is bounded by $b \cdot \exp(-\frac{1}{8}c^2)$. This can be seen from the inequality

$$|I_0(2|c x \sin(\hat{H})|) \exp(-\frac{1}{2}c^2 - x^2)| \\ \leq \exp(-\frac{1}{2}c^2 - x^2 + \frac{1}{2}|c|x) \\ \leq \exp(-\frac{1}{4}c^2 - \frac{3}{4}x^2). \quad (87)$$

(2) The contribution to (86) from the part $x < 1$ is bounded by $b \cdot \exp(-\frac{1}{4}c^2)$. This follows from

$$|I_0(2|c x \sin \Theta)| \exp\left(-\frac{1}{2}c^2 - x^2\right) \leq \exp\left(-\frac{1}{2}c^2 - x^2 + \frac{1}{2}|c|\right) . \quad (88)$$

Because of this, we can divide the domain of integration in (86) as $\mathcal{D}$ and $\mathcal{D}'$ where

$$\mathcal{D} = \{|\sin \Theta| > \frac{1}{4} \text{ and } x > 1\}$$

and $\mathcal{D}'$ is the complement of $\mathcal{D}$, and we can write for large $|c|$

$$|r_2(c, k)| \leq \int_{\mathcal{D}} d^3g dx \{ |g|^{-1} [|g|^{\gamma+x\gamma}] (|c|x)^{-\frac{1}{2}} e^{-Q} \} \\ + O[\exp(-\frac{1}{8}c^2)] , \quad (89)$$

$$Q = \frac{1}{2}c^2 + \frac{1}{2}\eta^2 + x^2 - 2x|c \sin \Theta| , \quad (90)$$

where the $\mathcal{D}'$ -contribution is written as $O[\exp(-\frac{1}{8}c^2)]$ on account of (87) and (88), and we have used large- c asymptotic estimate for $I_0(2|c x \sin \Theta|)$.

It is a simple matter to obtain from (90) the inequality

$$Q \geq \frac{1}{3}c^2 \cos^2 \Theta + \frac{1}{8}g^2 + (x - |c \sin \Theta|)^2 \quad (91)$$

which when substituted into (89) yields

$$|r_2(c, k)| \leq b(1+c^2)^{\frac{1}{2}\gamma-1} . \quad (92)$$

Again, we can use (75) and partially integrate $g_{\parallel}$, and the result is

$$|r_2(c, k)| \leq b(1+c^2)^{\frac{1}{2}\gamma-\frac{1}{2}} \left| \frac{1}{k} \right| . \quad (93)$$

Now combining (92) and (93) yields the final result for $r_2(c,k)$ as

$$|r_2(c,k)| \leq b(1+c^2)^{-\nu} (1+k^2)^{-\nu} . \quad (94)$$

It should be noted that for differentiation with respect to g , we have to consider the case $|\sin \theta| > \frac{1}{4}$ and $|\sin \theta| < \frac{1}{4}$ separately. For $|\sin \theta| > \frac{1}{4}$, we note that $|c \times g|$ can be differentiated with respect to g without difficulty; the differentiation only adds an extra factor of $|c|$ which we have in (93). On the other hand, for $|\sin \theta| < \frac{1}{4}$, we use

$$|\partial_g I_0(2|c \times g| \cot \theta)| \leq b(1+c^2 \cot^2 \theta)^{1/2} I_0(2|c \times g| \cot \theta). \quad (95)$$

Where the factor $(1+c^2 \cot^2 \theta)^{1/2}$ in (95) is covered by inequality (87).

Now to obtain estimates for the derivatives of $r_2(c,k)$ with respect to c and k we again have to consider $|\sin \theta| \geq \frac{1}{4}$ and $|\sin \theta| \leq \frac{1}{4}$ separately. For $|\sin \theta| \geq \frac{1}{4}$, we can differentiate $|c \times g|$ directly and for $|\sin \theta| \leq \frac{1}{4}$. We use

$$|\partial_c^m \partial_g I_0(2|c \times g| \cot \theta)| \leq b(1+c^2+g^2 \cot^2 \theta)^M |g|^{-1} I_0 , \quad (96)$$

where the factors in (96) are again covered by (87).

And the result can be obtained in the same way as the procedure for the estimate (94). In fact, we have

$$|\partial_c^m \partial_k^n r_2(c,k)| \leq b(1+c^2)^{-\nu} (1+k^2)^{-\nu} . \quad (97)$$

for $|m| \geq 0$, $|n| \geq 0$.

Summing the results for $r_1(c,k)$, $r_2(c,k)$ and $r_3(c,k)$ we have

$$|\partial_c^m \partial_k^n r(c,k)| \leq b(1+c^2)^{-\nu}(1+k^2)^{-\nu}, \quad (98)$$

for $|m| \geq 0$, $|n| \geq 0$.

(C) Estimates for $\zeta(c)q(c,k)$

Since $\zeta(c) \equiv 0$ for $|c| > 2c_0$, we only have to consider $q(c,k)$ for $|c| < 2c_0$. We first obtain from (17)

$$\begin{aligned} \text{Req}(c,k) \geq \exp(-4c_0^2 - 1) \int_{|g| \leq 1} d^3g |g|^\gamma \int_0^{\frac{1}{2}\pi} d\theta \beta(\theta, \alpha) \sin\theta (\sec\theta)^{3+\gamma} \\ [1 - \cos k \cdot g] \cdot e^{-g^2 \tan^2 \theta}. \end{aligned} \quad (99)$$

We now substitute (9) into (99) and note that the contribution due to $B(\tan\theta)$ in (9) gives a bounded function $b(c,k)$, i.e.

$$\begin{aligned} \text{Req}(c,k) \geq \exp(-4c_0^2 - 1) \int_{|g| \leq 1} d^3g \int_0^\infty dx \left(\frac{x}{g}\right)^{3-2\nu} e^{-x^2} \\ \times [1 - \cos k \cdot g] + b(c,k) \\ \geq k^{2\nu} \exp(-4c_0^2 - 1) \int_0^k dy y^{-1-2\nu} \left[\frac{\sin y}{y} - 1 \right] \\ + b(c,k) \end{aligned} \quad (100)$$

or, simply

$$\text{Req}(c,k) \geq b k^{2\nu} - \frac{1}{8} N \quad (101)$$

for $|c| < 2c_0$ if N is taken large enough.

The derivatives of $q(c,k)$ can be estimated likewise and the result is: for $|m| \geq 0$ we have

$$|\partial_c^m q(c,k)| \leq b(1+k^2)^v \quad (102)$$

and for $|m| \geq 0$ and $|n| \geq 1$, we have

$$|\partial_c^m \partial_k^n q(c,k)| \leq b(1+k^2)^{-v} \quad (103)$$

where both (102) and (103) are valid only for $|c| < 2c_0$.

(D) Estimates for $v(c,k)$

We finally come to the most important term $v(c,k)$. The procedure we use is somewhat similar to that for the estimates of $r_2(c,k)$ in the subsection (B).

We first write (24) out explicitly as

$$v(c,k) = \int d^3g (1 - e^{ik \cdot g}) \int_0^{\frac{1}{2}\pi} d\theta \zeta(\theta) G(\tan\theta) \\ \times [|g \tan\theta|^\gamma - |c|^\gamma] \exp\left\{-\frac{1}{2}c^2 - \frac{1}{2}n^2\right\}, \quad (104)$$

$$\zeta(\theta) = (\sin\theta)^{4v} (\sec\theta)^3 \beta(\theta, \alpha), \quad (105)$$

where $G(\tan\theta)$ is given in (7).

We then introduce the variable $x = g \tan\theta$ and write

$$\zeta(\theta) d\theta = \tilde{\zeta}(x, |g|) dx \quad (106)$$

and from (9) we have the following bound

$$|\tilde{\zeta}(x, |g|)| \leq b |g|^{-1} \left[1 + (x/|g|)^{2+2\nu} \right]. \quad (107)$$

Substituting (107) into (104) yields

$$|v(c, k)| \leq b \int d^3g |1 - e^{ik \cdot g}| \int_0^\infty dx |g|^{-1} \left[1 + (x/|g|)^{2+2\nu} \right] \\ \times \left| (x^\gamma - |c|^\gamma) \right| I_0(2|c|x \sin \Theta) \exp\left(-\frac{1}{2}c^2 - \frac{1}{2}\eta^2 - x^2\right) \quad (108)$$

Again we observe that contribution to (108) due to the parts of $|\sin \Theta| < \frac{1}{4}$ and $x < 1$ are bounded by $\exp(-\frac{1}{8}c^2)$; the argument is the same as those given in (87) and (88). Consequently, we only have to consider the domain $\mathcal{D}$ used in (B):

$$\mathcal{D} = \{ |\sin \Theta| > \frac{1}{4}, \quad x > 1 \},$$

where for large c the function I_0 can be asymptotically expanded; the result is

$$|v(c, k)| \leq b \int_{\mathcal{D}} d^3g dx \left\{ |1 - e^{ik \cdot g}| |g|^{-1} \left[1 + (x/|g|)^{2+2\nu} \right] \right. \\ \times \left| (x^\gamma - |c|^\gamma) \right| (|c|x)^{-\frac{1}{2}} e^{-Q} \Big\} \\ + O\left[k^{2\nu} \exp\left(-\frac{1}{8}c^2\right)\right] \quad (109)$$

where Q satisfies

$$Q = \frac{1}{2}c^2 + \frac{1}{2}\eta^2 + x^2 - 2x|c \sin \Theta| \\ \geq \frac{1}{3}c^2 \cos^2 \Theta + \frac{1}{8}g^2 + (x - |c \sin \Theta|)^2 \quad (110)$$

Now for large $|c|$, we observe that Q is large unless $\cos \Theta \approx 0$, $\sin \Theta \approx 1$ and $x \approx |c|$. We therefore collect a factor $|c|^{-1}$ from the integration $\sin \Theta d\Theta$, a $|c|^{-1}$ factor from $(|c|x)^{-\frac{1}{2}}$, a factor $|c|^{\gamma-1}$ from $|x^\gamma - |c|^\gamma|$ and a factor $|c|^{2+2\nu}$ from $x^{2+2\nu}$. The net result for (109) is

$$|v(c,k)| \leq b(1+c^2)^{-\nu}[k^2+1]^\nu \quad (111)$$

which is the desired result. We can make the above computation more rigorous by using $y = |c|\cos \Theta$, $t = x \sin \Theta$ as new integration variables in (109) and expand for large c ; the details are straightforward but tedious and we shall not reproduce them here.

To obtain estimates for the derivatives of $v(c,k)$ with respect to c , we again have to consider the cases $|\sin \Theta| < \frac{1}{4}$ and $|\sin \Theta| > \frac{1}{4}$ separately. For $|\sin \Theta| < \frac{1}{4}$, we use

$$|\partial_c^m I_0(2|cx\sin \Theta|)| \leq b(1+c^2+x^2)^M I_0(2|cx\sin \Theta|) \quad (112)$$

and the factor $(1+c^2+x^2)^M$ is covered by the factor $\exp(-\frac{1}{8}c^2)$ in (109).

For $|\sin \Theta| > \frac{1}{4}$, the procedure goes as follows.

First consider the term

$$\begin{aligned} Z &= I_0(2|cx\sin \Theta|) \exp(-\frac{1}{2}c^2 - \frac{1}{2}x^2 - x^2) \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-\Omega} d\phi \end{aligned} \quad (113)$$

in eqn. (108) where

$$\begin{aligned}
 \Omega &= \frac{1}{2}c^2 + \frac{1}{2}\eta^2 + x^2 - 2cx \sin(\mathbb{H}) \cos\phi \\
 &= (c \cos(\mathbb{H}) + \frac{1}{2}g)^2 + \frac{1}{4}g^2 + (x - |c \sin(\mathbb{H})| \cos\phi)^2 \\
 &\quad + c^2 \sin^2(\mathbb{H}) \sin^2\phi .
 \end{aligned} \tag{114}$$

We then differentiate with respect to c and obtain

$$\begin{aligned}
 \frac{1}{2}|\partial_c \Omega| &\leq |(c \cos(\mathbb{H}) + \frac{1}{2}g)| + |(x - |c \sin(\mathbb{H})| \cos\phi)| \\
 &\quad + |c \sin(\mathbb{H}) \sin\phi|
 \end{aligned} \tag{115}$$

which shows that

$$\begin{aligned}
 |\partial_c Z| &\leq \frac{b}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}\Omega} d\phi \\
 &\leq b I_0(|c x \sin(\mathbb{H})|) \exp(-\frac{1}{4}c^2 - \frac{1}{4}\eta^2 - \frac{1}{2}x^2) .
 \end{aligned} \tag{116}$$

On the other hand, since $|\sin(\mathbb{H})| > \frac{1}{4}$, repeated differentiations, which give factors like $|\sin(\mathbb{H})|^{-n}$, cause no problem. And a similar argument as that given above shows that

$$|\partial_c^m Z| \leq b I_0(|c x \sin(\mathbb{H})|) \exp(-\frac{1}{4}c^2 - \frac{1}{4}\eta^2 - \frac{1}{2}x^2) . \tag{117}$$

Now differentiate $v(c, k)$ in (104) and using (117), we obtain the same type of integrals as (108); in fact we would have exactly the same integral if we rescale c into $\sqrt{2} c$, η into $\sqrt{2} \eta$ and x into $\sqrt{2} x$. We therefore arrive at the same estimate as given in (111), namely

$$|\partial_c^m v(c,k)| \leq b(1+k^2)^{+v}(1+c^2)^{-v} \quad (118)$$

for $|m| \geq 0$.

Finally, we consider differentiation of $v(c,k)$ with respect to k . Since differentiation with respect to k_i only brings down a factor g_i from $(1-e^{ik \cdot g})$, the same procedure as in deriving (118) goes through without change and we have

$$|\partial_c^m \partial_k^n v(c,k)| \leq b(1+c^2)^{-v} \quad (119)$$

for $|n| \geq 1$ where we do not have the factor k^{2v} because of the presence of the extra factor g_i .

But on the other hand, since the k -differentiation brings an extra factor g_i , we can then use (75) and integrate by parts with respect to $g_{||}$. In doing this, we shall raise the power of $|c|$ by 1, which comes from differentiating Ω in (114) with $g_{||}$, and the result is

$$|\partial_c^m \partial_k^n v(c,k)| \leq b|k|^{-1}(1+c^2)^{-v+\frac{1}{2}}(1+k^2)^v \quad (120)$$

where $|n| \geq 1$. Now we combine (119) and (120) to obtain the desired result

$$|\partial_c^m \partial_k^n v(c,k)| \leq b(1+k^2)^{-v_1}(1+c^2)^{-v_1} \quad (121)$$

$$v_1 = v \frac{(1-2v)}{(1+2v)}. \quad (122)$$

for $|n| \geq 1$.

(E) Estimates for $s_\ell(c, k)$

We write (34) as

$$s_\ell(c, k) = |c|^\gamma \int_0^{\frac{1}{2}\pi} \beta(\theta, \alpha) (\sin \theta)^{4\nu} (2t)^{\frac{3}{2}} \exp\left(-\frac{1}{2}t \cos^2 \theta c^2\right) \\ \times \left[1 - \exp\left(-\frac{1}{2}tk^2 \cos^2 \theta\right)\right] d\theta + b(c, k) \quad (123)$$

where the function $b(c, k)$ and its derivatives with respect to c and k are bounded functions for all c and k . If we use $tk^2 \cos^2 \theta$ as a new integration variable for the integral in (123), we find that

$$\text{Res}_\ell(c, k) > b|c|^\gamma k^{2\nu} \exp(-c^2/k^2) - \frac{1}{8}N \quad (124)$$

if we take N large enough.

On the other hand, differentiations with c and k bring down extra factor of $\cos^2 \theta$, and consequently we have

$$|\partial_c^m s_\ell(c, k)| < b \quad |m| \geq 1 \quad (125)$$

$$|\partial_c^m \partial_k^n s_\ell(c, k)| < b(1+k^2)^{-\nu} \quad |n| \geq 1 \quad (126)$$

(F) Estimates for $\zeta(c)s(c, k)$

First we can give a bound for $s(c, k)$ by writing (30) as

$$|s(c, k)| \leq 2|c|^\gamma \int_0^{\frac{1}{2}\pi} \beta^*(\theta) \Lambda d\theta \\ \Lambda = [1 - J_0(t \sin 2\theta |c \times k|)] \\ + |\{1 - \exp[-\frac{1}{2}t \cos^2 \theta (k^2 + 2ic \cdot k)]\}| \quad (127)$$

We can use $|c \times k| \cos \theta$ as the new integration variable in (127) for the first term of Λ and $\cos \theta |k|$ for the second bracketed term in Λ . Since the singularity of $\beta^*(\theta)$ at $\theta = \frac{1}{2}\pi$ is given by $(\cos \theta)^{-1-2\nu}$, we obtain from (126):

$$\zeta(c) |s(c, k)| \leq b |k|^{2\nu} \quad (128)$$

This shows that if ε in (37) is small enough, then

$$\zeta(c) \operatorname{Re}[q(c, k) - \varepsilon s(c, k)] \geq \frac{1}{2} b |k|^{2\nu} - \frac{1}{8} N \quad (129)$$

which follows from (101).

Now for the derivatives of $s(c, k)$ with respect to c and k , we use the relation

$$\left| \left(\frac{d}{x} \right)^m J_0(x) \right| = |x^{-m} J_m(x)| \leq b \quad (130)$$

where x denotes $t \cdot \sin 2\theta |c \times k|$. We note that the derivatives of x^2 with respect to c and k can be easily calculated. The end result may be stated as

$$|\partial_c^m [\zeta(c) s(c, k)]| \leq b (1+k^2)^\nu, \quad (131)$$

$$|\partial_c^m \partial_k^n [\zeta(c) s(c, k)]| \leq b (1+k^2)^{-\nu}. \quad (132)$$

(G) Summary

We now use (37) and rewrite the symbol $a_\ell(c, k)$ as

$$\begin{aligned} a_\ell(c, k) = & \varepsilon s_\ell(c, k) + \zeta(c) [q(c, k) - \varepsilon s(c, k)] \\ & + [1 - \zeta(c)] [(1-\varepsilon) s_\ell(c, k) + v(c, k)] \\ & + r(c, k) + u(c) \end{aligned} \quad (133)$$

The results in the preceding subsections (A) - (F) may be summarized as follows.

For $c^2 < k^2$, we obtain from (124)

$$\operatorname{Re}[(1-\varepsilon) s_\ell(c, k)] > b |c|^\gamma |k|^{2\nu} - \frac{1}{8} N \quad (134)$$

and for $c^2 > k^2$, inequality (65) indicates that

$$u(c) > \frac{1}{2}b|c|^\gamma|k|^{2\nu} + \frac{1}{2}b|c|^{\gamma+2\nu} - \frac{1}{8}N \quad (135)$$

which together with (134) yield

$$\operatorname{Re}[(1-\epsilon)s_\ell(c,k) + \frac{1}{2}u(c)] > b(c^2+k^2)^\nu - \frac{1}{8}N \quad (136)$$

for all c and k . We next consider (111) and note that if $|c|$ is large enough, then

$$\operatorname{Re}[(1-\epsilon)s_\ell(c,k) + \frac{1}{2}u(c) + v(c,k)] > b(c^2+k^2)^\nu - \frac{1}{4}N \quad (137)$$

and we shall take c_0 to be so large that (137) is valid for $|c| > c_0$. We note that the choice of c_0 can be made independent of ℓ .

On the other hand, for $|c| < 2c_0$, we can use (129). The end result is

$$\begin{aligned} \operatorname{Re} a_\ell(c,k) &> \operatorname{Re}[\epsilon s(c,k) + \frac{1}{2}u(c)] \\ &+ N + b(1+c^2)^{\frac{1}{2}\gamma}(c^2+k^2)^\nu \end{aligned} \quad (138)$$

for all c and k .

We then note that by using the formula of Hilb for $P_\ell(\sin \theta)$ in $s_\ell(c,k)$, we can show that

$$\operatorname{Re}[\epsilon s_\ell(c,k) + \frac{1}{2}u(c)] > b(\ell+\frac{1}{2})^{2\nu} - \frac{1}{8}N \quad (139)$$

for all c and k , and its proof is similar to the arguments used in Sec. 5 of Part I. Combination of (138) and (139) yields (42).

Finally it is noted that (43) follows from (70), (98), (102), (118), (125) and (131) while (44) follows from (98), (103), (121), (126) and (132).

5. The Eigenvalue Problem

With the results in (42), (43) and (44) established, we are now in a position to invert the operator A_ℓ in (40). In fact we can introduce

$$B\phi = \int e^{ik \cdot c} [a_\ell(c, k)]^{-1} \hat{\phi}(k) d^3k \quad (140)$$

and show that

$$BA_\ell = 1 + R, \quad (141)$$

$$\|R\| < b/N, \quad (142)$$

where $\|R\|$ is the $\mathcal{L}^2$ -norm of R , and b independent of ℓ .

The proof of (141) and (142) is exactly the same as that given in Sec. 6 of Part I. From this result, it follows then

$$A_\ell^{-1} = (BA_\ell)^{-1}B \quad (143)$$

which implies that A_ℓ^{-1} is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$, since B is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$ according to Theorem 2 in Sec. 3 of Part I.

From (42) we also note that

$$|[a_\ell(c, k)]^{-1}| \rightarrow 0$$

as $\ell \rightarrow \infty$ and consequently we have, as $\ell \rightarrow \infty$,

$$\|A_\ell^{-1}\| \rightarrow 0. \quad (144)$$

We therefore arrive at the following conclusions.

(1) A^{-1} is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$.

(2) L^{-1} is a compact operator from $\mathcal{L}_1^2$ to $\mathcal{L}_1^2$, where $\mathcal{L}_1^2$ is the subspace of $\mathcal{L}^2$ which is orthogonal to the five eigenfunctions of L for the eigenvalue zero (i.e. orthogonal to the five collisional invariants with the proper weight factor).

(3) The spectrum of L is discrete.

(4) The eigenfunctions of L are all $\mathcal{L}^2$ -functions and they are complete in $\mathcal{L}^2$ since L is symmetric and non-negative and therefore essentially self-adjoint.

(5) The $\mathcal{L}^2$ -eigenfunctions of L in fact belong to $\mathcal{S}$; this will be proved in the next section.

6. Properties of the Eigenfunctions

We shall now show that the eigenfunctions ϕ of L in fact belong to $\mathcal{S}$; i.e. they are C^∞ -functions having a decay at infinity faster than any inverse powers of c .

We first show that the eigenfunctions ϕ are C^∞ -functions. To this end, we recall eqn. (40) and write

$$\lambda\phi = A_\ell\phi = Y\psi, \quad (145)$$

$$Y\psi = \int e^{ik \cdot c} a_\ell(c, k) (1+k^2)^{-\frac{1}{2}v} \hat{\psi}(k) dk, \quad (146)$$

$$\hat{\psi}(k) = (1+k^2)^{\frac{1}{2}v} \hat{\phi}(k). \quad (147)$$

Then we can invert Y in the same way as we invert A_ℓ and Y^{-1} is a bounded operator from $\mathcal{L}^2$ to $\mathcal{L}^2$. Consequently, since $\phi \in \mathcal{L}^2$,

$$\psi = \lambda Y^{-1} \phi$$

shows that $\psi \in \mathcal{L}^2$ or $(1+k^2)^{\frac{1}{2}v} \hat{\phi}(k) \in \mathcal{L}^2$.

Next we write

$$\psi = X\phi = \int e^{ik \cdot c} (1+k^2)^{\frac{1}{2}v} \hat{\phi}(k) dk \quad (148)$$

and apply the operator X to (145). This gives

$$\lambda\psi = YX\psi + [X, Y]\psi \quad (149)$$

where $[X, Y]$ is the commutator of X and Y .

Now applying Y^{-1} to the last equation, we obtain

$$X\psi = \lambda Y^{-1}\psi - Y^{-1}[X, Y]\psi \quad (150)$$

It can be shown that the operator $Y^{-1}[X, Y]$ is a bounded operator from $\mathcal{L}^2$ to $\mathcal{L}^2$; the proof is similar to the procedure in Sec. 6 of Part I using the reverse operator and multiple symbols and we shall not give it here. The conclusion is then $X\psi \in \mathcal{L}^2$ or $(1+k^2)^v \hat{\phi}(k) \in \mathcal{L}^2$.

Proceeding in this manner with mathematical induction leads to the result that

$$(1+k^2)^{\frac{1}{2}nv} \hat{\phi}(k) \in \mathcal{L}^2 \quad (151)$$

for all positive integer n . This shows that $\phi(c)$ is a C^∞ -function.

In this proof, the main property we use is the growth $(1+k^2)^v$ in k in $a_\ell(c, k)$ in (42). Since $a_\ell(c, k)$ also has a growth $(1+c^2)^v$ in c , we can use similar arguments and show that

$$(1+c^2)^{\frac{1}{2}nv} \phi(c) \in \mathcal{L}^2, \quad (152)$$

but the proof is slightly more complicated than that for (151) and will not be reproduced here.

It is further mentioned if we use a mixture of the arguments used in establishing (151) and (152) we can show that $(1+c^2)^{nv/2} \partial_c^m \phi$ belongs to $\mathcal{L}^2$; i.e. $\phi \in \mathcal{S}$.

7. Concluding Remarks

Finally we make a remark concerning the part of the collision integral conventionally written as

$$C_1[f] = \int d^3c_1 \exp(-c_1^2) \int d\varepsilon d\theta \sigma(|g|, \theta) [f'_1 - f_1]. \quad (153)$$

If we use $\phi(c) = f(c) \exp(-\frac{1}{2} c^2)$ then the operator L_1 ,

$$L_1\phi = \exp(-\frac{1}{2} c^2) C_1[\phi(c) \exp(\frac{1}{2} c^2)],$$

is a compact operator from $\mathcal{L}^2$ to $\mathcal{L}^2$, which can be seen from the estimates in (98) and Theorem 3 in Sec. 3 of Part I.

This shows that all the difficulty in the Boltzmann Collision operator stems from the term involving $[f'_1 - f_1]$.

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